Update on Diophantine records for genus-3 curves

ICERM workshop on

Computational Aspects of L-functions

and related topics

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REVIEW:
Mordell-Faltings and Caporaso-Harris-Mazur

Theorem (Faltings 1983, conjectured by Mordell c.1920): *Let C be an algebraic curve of genus $g > 1$ over a number field K. Then $|C(K)| < \infty$.*

Now fix K and $g > 1$, and vary C. Can the number $|C(K)| < \infty$ get arbitrarily large? In other words: Is

$$B(g, K) := \sup_{C} |C(K)|$$

infinite?

Theorem (Caporaso-Harris-Mazur 1997): *Assume Bombieri-Lang conjecture. Then $B(g, K) < \infty$ for all $g > 1$ and K.*

The Bombieri-Lang conjecture is an analogue of Mordell-Faltings for algebraic varieties of arbitrary dimension:

Conjecture (Bombieri-Lang 1986): *Suppose V is an algebraic variety of general type. Then all its rational points are in a finite union of subvarieties V_i' each of dimension $< \dim(V)$.*

So, assuming Bombieri-Lang, we have $B(g, K)$ by Caporaso-Harris-Mazur.

What's the Caporaso-Harris-Mazur bound on $B(g, K)$?

Alas the proof gives no explicit upper bound, because (as with Faltings) the argument is ineffective — as it must be: already for $\dim(V) = 1$ the exceptional V_i' are the rational points of the curve V, and in general we have no control over their number.

So, we don't have an upper bound on any $B(g, K)$, not even conditional on Bombieri-Lang.

The best we can do for now is try hard to find lower bounds on $B(g, K)$, usually by constructing curves of given genus $g > 1$ with numerous rational points.

Strategy: construct an infinite family of genus-g curves with many sections, which gives a lower bound on

$$N(g, K) := \limsup_{C} |C(K)|,$$

and then seek specializations with even more points.

The first case is $(g, K) = (2, \mathbf{Q})$. We review our results there, and then report on recent work for $B(3, \mathbf{Q})$ and $N(3, \mathbf{Q})$.

Since we assume $K = \mathbf{Q}$ henceforth, we use $B(g)$ and $N(g)$ to denote $B(g, \mathbf{Q})$ and $N(g, \mathbf{Q})$ respectively.

GENUS 2

Keller and Kulesz 1995: at least $12 \cdot 49 = 588$ points on

$$Y^2 = 278271081X^2(X^2-9)^2 - 229833600(X^2-1)^2;$$

but without extra symm.: at least $2 \cdot 187 = 374$ points on

$$Y^2 = 1306881X^6 + 18610236X^5 - 46135758X^4 - 1536521592X^3 \\ - 2095359287X^2 + 32447351356X + 89852477764.$$

(Stahlke 1997, including four point-pairs found later).

NDE + M.Stoll 2008–09: at least $2 \cdot 321$ points on

$$Y^2 = 82342800X^6 - 470135160X^5 + 52485681X^4 \\ + 2396040466X^3 + 567207969X^2 - 985905640X + 15740^2,$$

with X equal

$0, -1, -4, 4, 5, 6, 1/3, -5/3, -3/5, 7/4, \ldots, 148596731/35675865,$
$58018579/158830656, 208346440/37486601,$
$-1455780835/761431834, -3898675687/2462651894.$

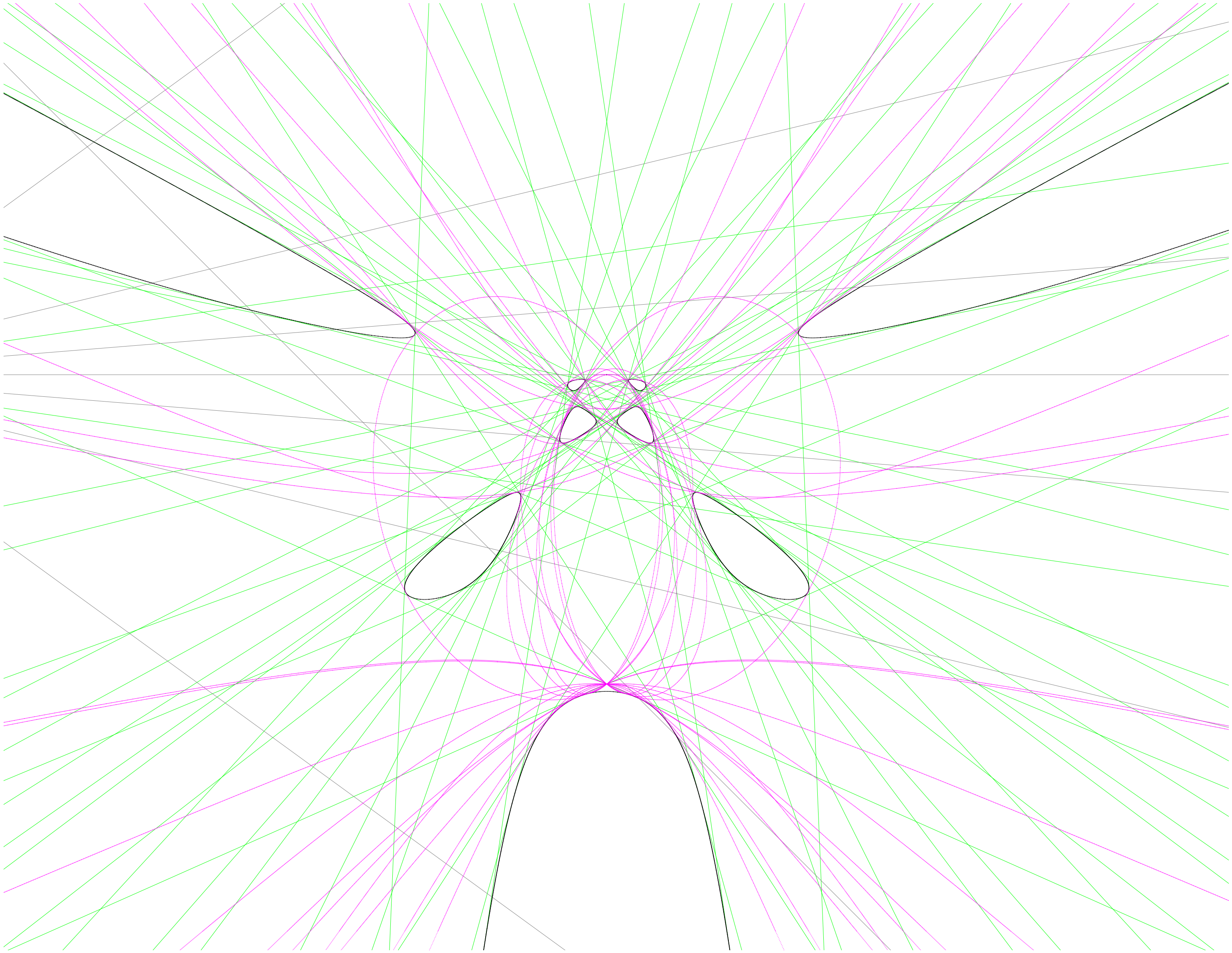
So $B(2) \geq 642$. Also $N(2) \geq 150$ (at least twice the previous record).

For curves of genus 2 with a rational Weierstrass point (equivalently, curves $y^2 =$ quintic), "$B_W(2)$" is at least 303, from

$$Y^2 = 98017920X^5 - 3192575X^4 - 274306650X^3 + 256343425X^2 - 76075320X + 2740^2,$$

and "$N_W(2)$" is at least $1 + 2 \cdot 59 = 119$.

Where did these new records come from? Pictures such as



This is a “double plane” model of the K3 surface X with $\mathrm{NS}_{\mathbf{Q}}(X)$ of rank 20 (maximal) and discriminant -163.

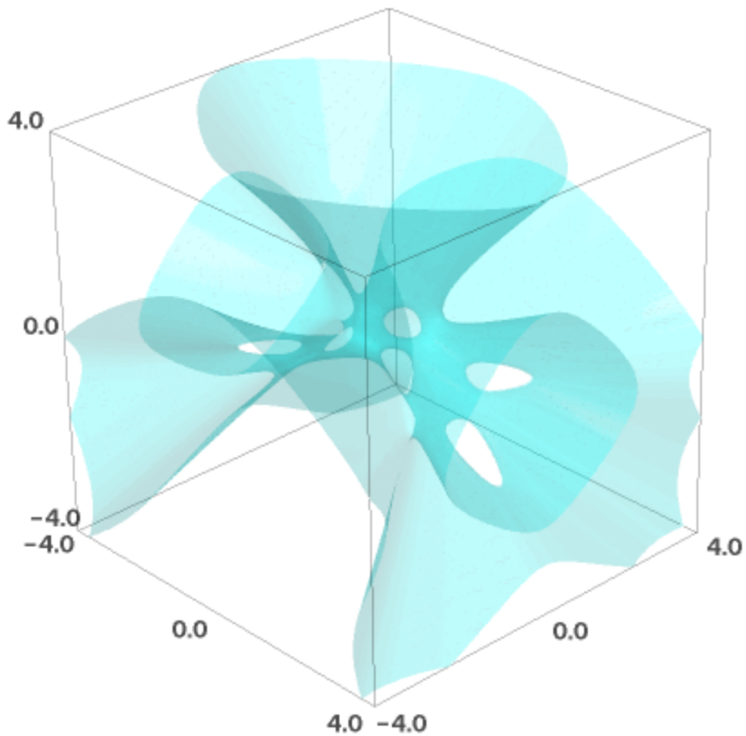
My first excuse for presenting this at an L-function workshop: the L-function of $H^2(X)$ is one way to prove that if $\mathrm{rank}(\mathrm{NS}_{\mathbf{Q}}(X)) = 20$ then $\mathrm{disc}(\mathrm{NS}_{\mathbf{Q}}(X))$ is one of the 13 negative discriminants of quadratic fields of class number 1. [This is Schütt’s proof; NDE: alternative proof by reconstructing the elliptic curve E such that $X \sim \mathrm{Km}(E \times E)$.]

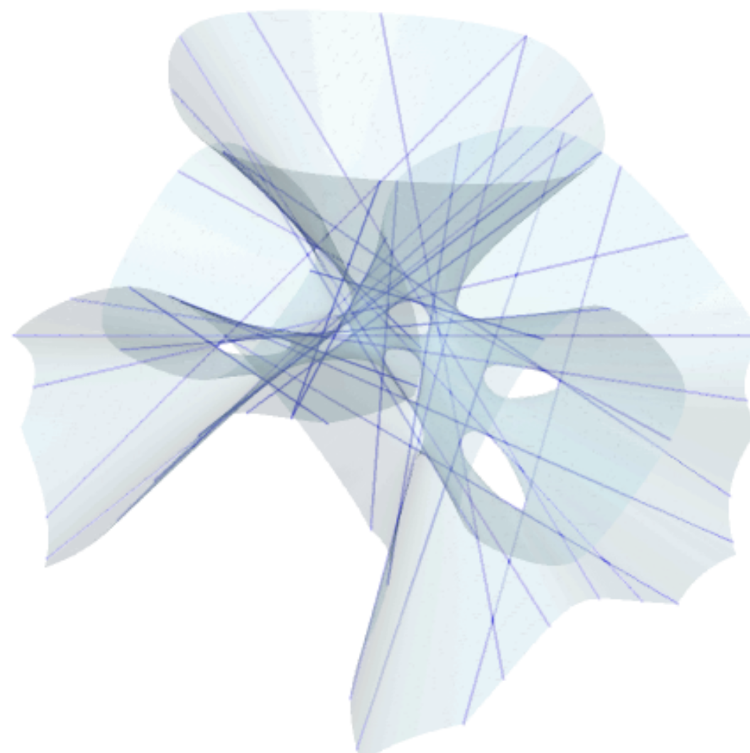
Double plane $\Rightarrow$ ample H with $H \cdot H = 2$, so $H^\perp = L\langle -1 \rangle$ for some pos.-def. even lattice L of discriminant $2\,|\mathrm{disc}\,\mathrm{NS}(X)|$; tritangent lines $\longleftrightarrow$ pairs of half-lattice vectors of norm $5/2$ modulo root lattice $R(L)$. So want $R(L)$ small, and even for $\mathrm{disc}(L) = -163$ barely get $R(L) = A_1$.

6+ YEARS LATER: GENUS 3

Two possibilities: hyperelliptic or quartic. So, split $B(3)$ and $N(3)$ questions into $B_H(3)$ and $B_Q(3)$ (resp. $N_H(3)$ and $N_Q(3)$).

For quartics, use sections of quartic surface with many lines, e.g.





Pretty pictures, but lots of choices — over 10^3 quartic models of the CM K3 of discriminant -163 (rank-19 even lattices of disc. $4\cdot 163$ with minimum 4), and that might not even be the best place to look. Also, it's not that easy to search for points on a given quartic curve.

So, started with hyperelliptic genus-3 curves, as did previous work, again by Stahlke ($2\cdot 52 = 104$, generic symmetry) and Keller-Kulesz ($16\cdot 11 = 176$):

$$\begin{aligned} Y^2 = {} & 76X^8 + 671X^7 - 8539X^6 - 89512X^5 + 147851X^4 \\ & + 3076727X^3 + 6159667X^2 - 3720486X - 3527271, \end{aligned}$$

$$Y^2 = 7920000(X^2+1)^4 - 136782591X^2(X^2-1)^2.$$

(all of height $< 10^3$).

Here, instead of plane slices of a quartic surface, we search among preimages of a $(1,1)$ curve on a double cover

$$u^2 = P(s,t)$$

of $\mathbf{P}^1 \times \mathbf{P}^1$ branched at a smooth $(4,4)$ curve $P(s,t) = 0$. Now there are only two such models of our -163 surface; one has $s \leftrightarrow t$ symmetry, and 5 of the 8 points on the $s = t$ axis are rational, so we even have a source of "$B_{HW}(3)$" curves (hyperelliptic with Weierstrass point), among which

$$\begin{aligned} Y^2 = 18869760X^7 &+ 295557444X^6 + 1638491652X^5 \\ &+ 4305582969X^4 + 5721193700X^3 \\ &+ 3648196500X^2 + 895300000X + 7000^2 \end{aligned}$$

has $|C(\mathbf{Q})| \geq 1 + 2 \cdot 70 = 141$: Weierstrass $x = \infty$ and pairs at $x = 0, 2, -2/3, -3/2, 4/3, -5/2, \ldots, -3409/1500, -28727/20886$.

Without the Weierstrass-point restriction, we tie the overall record of Keller-Kulesz for $B(3)$ by finding a hyperelliptic curve with $|C(\mathbf{Q})| \geq 2 \cdot 88 = 176$ and no extra symmetry:

$$\begin{aligned} Y^2 = {} & 5780865024X^8 - 88857648000X^7 + 542817272736X^6 \\ & - 1616473139664X^5 + 2143113743265X^4 - 145305843468X^3 \\ & - 2058755904906X^2 + 363486538980X + 1262256306129; \end{aligned}$$

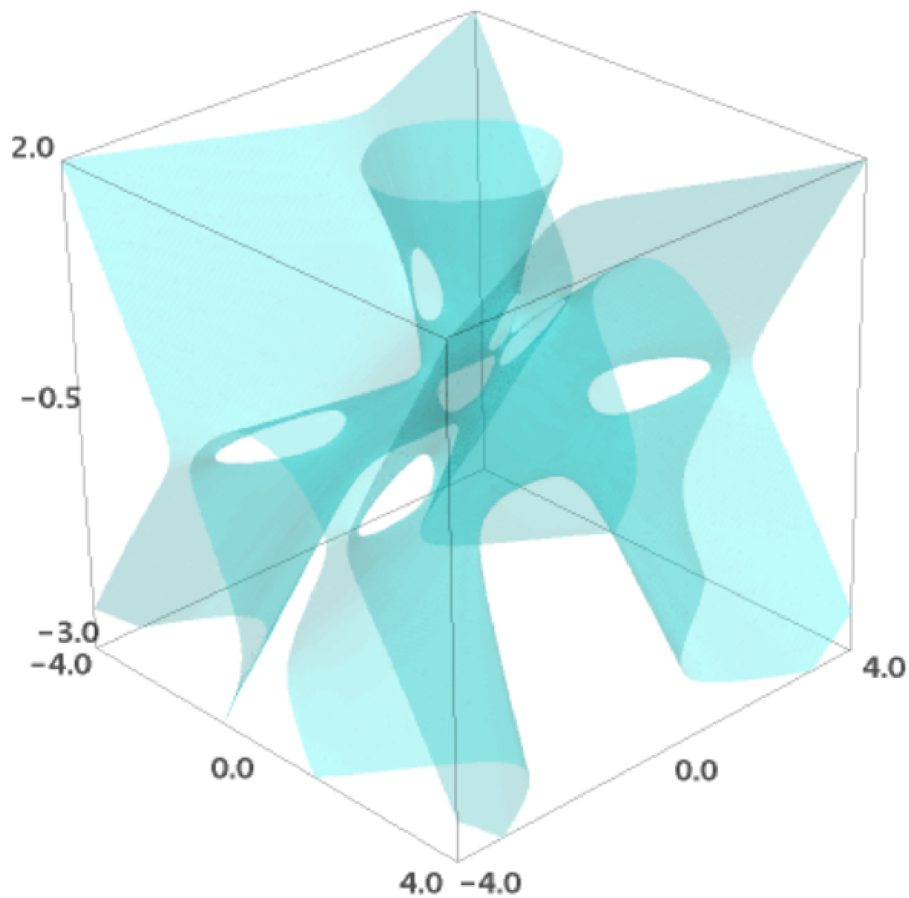
searching (with M. Stoll's **ratpoints**, as earlier) up to height $2^{20} = 10^{6+\epsilon}$ finds rational points with

$$x = \infty, -1, -2, -2/3, 1/3, 3, 3/2, -4, \ldots, 40177/17204,\ 54317/22468.$$

We also come close to the Kulesz bound $N_H(3) \geq 72$ (a la Mestre, though again that uses curves with 16 symmetries, so 5 orbits, and ours have only $\{1, \iota\}$ and 30+ orbits).

A few months ago: back to quartics. The largest number of lines on a quartic model of X seems to be 46 (possibly the $\mathbf{Q}$-max over all quartic surfaces), so we immediately get $N_Q(3) \geq 46$; this already breaks the previous record of 37 (Kulesz 1998), and we can still play with conics on X etc.

Current record for $N_Q(3)$ is 64.



For $B_Q(3)$ we expect to find slices with many more points than just the line count, so trying lots of quartic models may be better strategy than concentrating on the one with most lines. For now, the record comes from one of the 42-line models: at least $2 \cdot 72 = 144$ rational points on

$$
\begin{aligned}
&4x^4 - (37y^2 + 67yz + 13586z^2)x^2 \\
&\quad + (9y^4 + 4383y^3z + 75814y^2z^2 - 1819700yz^3 - 12562100z^4) \\
&\quad = 0,
\end{aligned}
$$

ranging from $(\pm 1 : 2 : 0)$ and $(\pm 3 : 1 : 0)$ to $(\pm 3844461 : 1799015 : 52173)$ and $(\pm 26758059 : -3088913 : 447931)$.

Second excuse for presenting this at an L-function workshop: The same families that produce curves of genus 2 and 3 with many rational points also produce *simple* genus-2 and genus-3 Jacobians of record Mordell–Weil rank r. We find $r \geq 29$ for

$$\begin{aligned} Y^2 = {} & 3115323179136X^6 + 13377846720672X^5 \\ & + 2083591459177X^4 - 31185870903704X^3 \\ & + 3365838909904X^2 + 11170486506240X + 1337760^2, \end{aligned}$$

and $r \geq 31$ for

$$\begin{aligned} Y^2 = {} & 3690^2X^8 + 136193480460X^7 + 855554427369X^6 \\ & - 973414777968X^5 + 8046400145942X^4 + 7241370511844X^3 \\ & + 2187498173777X^2 + 273643583472X + 110152^2, \end{aligned}$$

in each case generated by points of height $< 10^3$.

[Without requiring *simple* Jacobians, can "cheat" with known high-rank elliptic curves; e.g. $J(C) = (E \times E)/(2,2)$ of rank 38 for an elliptic curve E with MW group $(\mathbf{Z}/2\mathbf{Z}) \otimes \mathbf{Z}^{19}$.]

What about quartics? Here I don't know because it's harder to compute the rank of the Jacobian generated by a given list of divisors.

For hyperelliptic, $y^2 = P(x)$, we adapt a suggestion of Nils Bruin: Let p be a large prime and r_1, r_2 mod p roots of P; then

$$J(\mathbf{Q})/2J(\mathbf{Q}) \to \mathbf{Z}/2\mathbf{Z}, \quad (x,y) = \chi_p\left(\frac{x-r_1}{x-r_2}\right)$$

is a homomorphism [with $\chi_p = (\cdot/p)$]. Try N such primes (I used $N = 48$), get $J(\mathbf{Q})/2J(\mathbf{Q}) \to (\mathbf{Z}/2\mathbf{Z})^N$; dim(image) is lower bound on rank (assuming $J(\mathbf{Q})[2] = \{0\}$). If also P irreducible and $P = a^2x^{2g+2} + O(x^{2g+1})$, raise bound by 1 (N. Bruin says it's an example of the "Cassels kernel").

But no such simple recipe for a quartic curve. Any ideas?

THE END

THANK YOU.

Any (more) questions?

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